Value at Risk application to FSD portfolio efficiency testing

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Abstract

The paper deals with efficiency testing of a given portfolio with respect to all other portfolios that can be created from the considered set of assets. The efficiency is based on the first order stochastic dominance (FSD) relation. A necessary and sufficient condition for the first order stochastic dominance criterion is expressed in terms of Value at Risks (VaRs). Consequently a FSD portfolio efficiency test based on VaRs is formulated. Contrary to the usual case, a general discrete distribution of portfolio returns is assumed what makes the test computationally more demanding comparing to the equiprobable scenarios case. Therefore we present a tractable reformulation of this test that turns constraints on VaRs into classical mixed-integer nonlinear programming problem.

Key words

Value at Risk, first order stochastic dominance, portfolio efficiency

JEL Classification: D81, G11

1. Introduction

The basics of decision-making theory were presented in the seminal work of Harry Markowitz (1952). He identified two main components of portfolio performance, mean return and risk represented by variance. Applying a simple parametric optimization model (mean-risk model) one can find the efficient frontier. In this case, the portfolio is seen as efficient if there is no better portfolio, i.e., a portfolio with a higher mean and smaller variance. Later on, the theory of mean-risk models has been enriched by other risk measures, for example, semivariance, see Markowitz (1959), Value at Risk (VaR) or Conditional Value at Risk (CVaR), see Pflug (2000), Rockafellar and Uryasev (2002).

Alternatively, a decision-making rules may be based on utility functions (von Neumann, Morgenstern (1944)) instead of risk measures. It leads to maximizing expected utility problems where investor's risk attitude is described by an utility function. If the utility function is perfectly known, one can find the optimal decision. If that is not the case, one can at least identify the set of efficient portfolios with respect to all utility functions, that is, assuming only non-satiation for the investor's preferences. It leads to the first-order stochastic dominance (FSD) relation (see Levy (2006) and references therein). Applying this relation, a given portfolio is classified as FSD efficient if there is no other FSD dominating portfolio. On the other hand, a given portfolio is FSD inefficient if a FSD dominating portfolio exists. The FSD dominating portfolio has the following property: every non-satiable decision maker prefers it to the given portfolio. Under assumption of equiprobable discrete distribution, Kuosmanen (2004) presented a mixed-integer program for testing whether a given portfolio is FSD efficient or not. An alternative approach to FSD efficiency was introduced in Kopa and

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Post (2009) where a given portfolio is classified as FSD efficient (optimal) if it is an optimal solution of the maximizing expected utility problem for at least one utility function.

Simultaneously, the second order stochastic dominance relation (SSD) and its application to portfolio efficiency were extensively studied in the last ten years. The second order stochastic dominance criterion offers a relevant tool for portfolio efficiency analysis if we consider only risk-averse and risk neutral decision makers. The corresponding portfolio efficiency tests were introduced and applied to finance data in e.g. Post (2003), Kuosmanen (2004), Kopa and Chovanec (2008), Kopa (2010), Dupačová and Kopa (2011), Branda and Kopa (2012), and Lizyayev (2012).

In this paper we limit our attention to portfolio efficiency testing with respect to the first order stochastic dominance. We follow Kuosmanen (2004) in the FSD efficiency definition. However, contrary to Kuosmanen (2004) we express the stochastic dominance criterion in terms of Value at Risks. Moreover, we assume a general discrete distribution of returns rather than the equiprobable scenarios as in Kuosmanen (2004). Firstly, we recall a general expression of FSD relation and then we apply it to the necessary and sufficient condition formulation. Finally, we reformulate it in the form of a mixed-integer nonlinear program that is computationally more tractable.

The remainder of this paper is structured as follows. Section 2 presents, basic notations, definitions and formulations of FSD relation. It focuses on the VaR expressions. It is followed by the necessary and sufficient condition for FSD portfolio efficiency. Section 4 presents a tractable reformulation of FSD portfolio efficiency tests.

2. Preliminaries

Let us consider a random vector $\mathbf{r} = (r_1, r_2, \dots, r_n)'$ of returns of n assets with discrete probability distribution that takes T scenarios with probabilities $\mathbf{p} = (p_1, p_2, \dots\ p_T)$. The returns of the assets for the various scenarios are given by

$$X = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^T \end{pmatrix}$$

where $\mathbf{x}^t = (x_1^t, x_2^t, \dots, x_n^t)$ is the t-th row of matrix X. We will use $\mathbf{v} = (v_1, v_2, \dots, v_n)'$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)'$ for the vectors of portfolio weights. Throughout the paper, we will consider a compact convex set of portfolio weights W.

For any portfolio $\mathbf{v} \in W$, let $(-X\mathbf{v})^{[k]}$ be the k-th smallest element of $(-X\mathbf{v})$, i.e.

$(-X\mathbf{v})^{[1]} \leq (-X\mathbf{v})^{[2]} \leq \dots \leq (-X\mathbf{v})^{[T]}$ and let $I(\mathbf{v})$ be a permutation of the index set $I = (1,2, \dots, T)$ such that $-\mathbf{x}^{i(\mathbf{v})}\mathbf{v} = (-X\mathbf{v})^{[i]}$ for all $i \in I$. Accordingly, we can order the corresponding probabilities and we denote $p_i^{\mathbf{v}} = p_{i(\mathbf{v})}$. Hence, $p_i^{\mathbf{v}} = \mathrm{P}\left(-\mathbf{rv} = (-X\mathbf{v})^{[i]}\right)$.

Moreover, we consider cumulative probabilities: $q_s^{\mathbf{v}} = \sum_{i=1}^{s} p_i^{\mathbf{v}},\ s = 1,2, \dots, T$. The same notation is applied for the tested portfolio $\mathbf{w} \in W$.

Under these assumptions and notations, the general definition of Value at Risk of portfolio $\mathbf{w} \in W$ at level α can be reformulated as follows.

Definition 1:

Let s be the index satisfying $q_{s-1}^{\mathbf{v}} < \alpha \leq q_s^{\mathbf{v}}$. Then:

$$\mathrm{VaR}_\alpha(-\mathbf{rw}) = (-X\mathbf{w})^{[s]}.$$

For $\alpha = 0$ we define: $\mathrm{VaR}_0(-\mathbf{rw}) = (-X\mathbf{w})^{[1]}$ in order to have $\mathrm{VaR}_\alpha(-\mathbf{rw})$ (as a function of α) right continuous in zero. We start our efficiency analysis with definition of pair-wise comparison between two portfolios.

Definition 2:
We say that a portfolio $\mathbf{v} \in W$ dominates another portfolio $\mathbf{w} \in W$ with respect to the first order stochastic dominance if:

$\mathrm{P}(-\mathbf{rv} \le \theta) \le \mathrm{P}(-\mathbf{rw} \le \theta)$ for all $\theta \in \mathrm{R}$ with strict inequality for at least one $\theta \in \mathrm{R}$.

In financial applications FSD relation allows us to incorporate random benchmarks (defined on the same probability space) instead of fixed thresholds. The FSD relation can be alternatively verified as follows:

- portfolio $\mathbf{v}$ FSD dominates portfolio $\mathbf{w}$ if and only if $Eu(\mathbf{rv}) \ge Eu(\mathbf{rw})$ for all utility functions u with strict inequality for at least some u provided the expected values above are finite, see for example Levy (2006)
- portfolio $\mathbf{v}$ FSD dominates portfolio $\mathbf{w}$ if and only if $\mathrm{VaR}_\alpha(-\mathbf{rv}) \le \mathrm{VaR}_\alpha(-\mathbf{rw})$ for all $\alpha \in [0,1]$ with strict inequality for at least some $\alpha \in [0,1]$, see e.g. Ogryczak and Ruszczynski (2002).

Since we limit our attention to a discrete probability distribution of returns, the inequality of VaRs need not be verified in all $\alpha \in [0,1]$, but only in at most $2S$ particular points, see Dupačová and Kopa (2012) for more details and the proof of the following result.

Theorem 1:
A portfolio $\mathbf{v}$ dominates portfolio $\mathbf{w}$ with respect to FSD if and only if $\mathrm{VaR}_{q_s^{\mathbf{v}}}(-\mathbf{rv}) \le \mathrm{VaR}_{q_s^{\mathbf{v}}}(-\mathbf{rw})$ and $\mathrm{VaR}_{q_s^{\mathbf{w}}}(-\mathbf{rv}) \le \mathrm{VaR}_{q_s^{\mathbf{w}}}(-\mathbf{rw})$ for all $s = 1,2,\dots,T$ with strict inequality for at least one $q_s^{\mathbf{v}}$ or $q_s^{\mathbf{w}}$.

3. FSD portfolio efficiency testing

In this section we follow Dupačová and Kopa (2012). We define FSD portfolio efficiency and present a test for FSD portfolio efficiency that focuses on searching for a dominating portfolio.

Definition 3:
A given portfolio $\mathbf{w} \in W$ is FSD inefficient if there exists portfolio $\mathbf{v} \in W$ such that $\mathbf{v}$ FSD dominates portfolio $\mathbf{w}$. Otherwise, portfolio $\mathbf{w} \in W$ is FSD efficient.

This definition classifies portfolio $\mathbf{w} \in W$ as FSD efficient if and only if no other portfolio is better (in the sense of the FSD relation) for all decision makers. Following Kopa and Post (2009), Definition 3 formulates FSD efficiency in the sense of "FSD admissibility". Alternatively, one may define it as "FSD optimality". See Kopa and Post (2009) for more details. In this paper we focus on efficiency approach based on Definition 3. In order to find a FSD dominating portfolio $\mathbf{v} \in W$, we may solve the following problem:

$$\varphi(\mathbf{w}) = \min_{a_s, b_s, \mathbf{v}} \sum_{s=1}^{T} a_s + b_s \tag{1}$$

$$\text{s.t.}$$

$$\mathrm{VaR}_{q_s^{\mathbf{v}}}(-\mathbf{rv}) - \mathrm{VaR}_{q_s^{\mathbf{v}}}(-\mathbf{rw}) \le a_s,\ s=1,2,\dots,T$$

$$\mathrm{VaR}_{q_s^{\mathbf{w}}}(-\mathbf{rv}) - \mathrm{VaR}_{q_s^{\mathbf{w}}}(-\mathbf{rw}) \leq b_s,\ s=1,2,\ldots,T$$

$$a_s, b_s \leq 0,\ s=1,2,\ldots,T$$

$$\mathbf{v} \in W$$

The objective function represents the sum of differences between VaRs of a portfolio $\mathbf{v} \in W$ and VaRs of the tested portfolio $\mathbf{w} \in W$. The differences are considered in points $q_s^{\mathbf{v}}$ and $q_s^{\mathbf{w}}$. The other points need not be taken into account, because VaR_α is a piecewise constant function in α. All differences must be non-positive and at least one negative to guarantee that portfolio $\mathbf{v}$ dominates portfolio $\mathbf{w}$. On the other hand, if no dominating portfolio exists, that is, portfolio $\mathbf{w}$ is FSD efficient, then $\varphi(\mathbf{w}) = 0$. Summarizing, Theorem 1 implies the following necessary and sufficient FSD portfolio efficiency test proved in Dupačová and Kopa (2012).

Theorem 2:
A given portfolio $\mathbf{w}$ is FSD efficient if and only if $\varphi(\mathbf{w}) = 0$. If $\varphi(\mathbf{w}) < 0$ then the optimal portfolio of (1) $\mathbf{v}^*$ is FSD efficient and it dominates portfolio $\mathbf{w}$ by FSD.

4. A tractable reformulation of FSD efficiency test

Although Theorem 2 provides a necessary and sufficient condition for FSD efficiency that can be used for testing, problem (1) cannot be directly solved. One should first rewrite VaR terms in a more tractable way. If the probabilities of scenarios are equal than (1) simplifies to:

$$\varphi(\mathbf{w}) = \min_{a_s,\mathbf{v}} \sum_{s=1}^{T} a_s \tag{2}$$
$$\text{s.t.}$$
$$\mathrm{VaR}_{\frac{s}{T}}(-\mathbf{rv}) - \mathrm{VaR}_{\frac{s}{T}}(-\mathbf{rw}) \leq a_s,\ s=1,2,\ldots,T$$

$$a_s \leq 0,\ s=1,2,\ldots,T$$

$$\mathbf{v} \in W$$

Applying Definition 1, the VaR constraints can be reformulated using permutation matrix what directly leads to the Kuosmanen test. Unfortunately, under assumption of general discrete distribution of returns, all four VaR terms: $\mathrm{VaR}_{q_s^{\mathbf{v}}}(-\mathbf{rv})$**,** $VaR_{q_s^{v}}(-rw)$, $\mathrm{VaR}_{q_s^{\mathbf{w}}}(-\mathbf{rv})$**,** $\mathrm{VaR}_{q_s^{\mathbf{w}}}(-\mathbf{rw})$ must be equivalently rewritten using auxiliary (integer) variables and constraints. Without loss of generality, prior to testing, we increasingly order the vectors of scenarios according to losses of the tested portfolio $\mathbf{w}$ such that: $-\mathbf{x}^i\mathbf{w} = (-X\mathbf{w})^{[i]}$ for all $i = 1,2,\ldots,T$. It allows simple calculation of $\mathrm{VaR}_{q_s^{\mathbf{w}}}(-\mathbf{rw})$ because: $\mathrm{VaR}_{q_s^{\mathbf{w}}}(-\mathbf{rw}) = -\mathbf{x}^i\mathbf{w}$. Moreover, the following theorem can be applied to $VaR_{q_s^{v}}(-rw)$ reformulation.

Theorem 3:
For any $\alpha \in [0,1]$ and any portfolio $\mathbf{w} \in W$:

$$VaR_\alpha(-rw) = -\mathbf{x}^1\mathbf{w}y_1^{w,\alpha} + \sum_{j=\mathbf{2}}^{T}(-\mathbf{x}^j\mathbf{w} + \mathbf{x}^{j-1}\mathbf{w})y_j^{w,\alpha}$$

where $\sum_{j=1}^{T} y_j^{w,\alpha} p_j \geq \alpha$, $\sum_{j=1}^{T} z_j^{w,\alpha} p_j \leq \alpha$ and $y_j^{w,\alpha}$**,** $z_j^{w,\alpha}$**,** $j = 1,2,\ldots,T$ are binary variables satisfying: $y_j^{w,\alpha} \geq y_{j-1}^{w,\alpha}$**,** $z_j^{w,\alpha} \geq z_{j-1}^{w,\alpha}$, $j = 2,\ldots,T$ and $\sum_{j=1}^{T} z_j^{w,\alpha} + 1 = \sum_{j=1}^{T} y_j^{w,\alpha}$.

Proof: The conditions on binary variables guarantee that the variables $y_j^{w,\alpha}$are equal to variables $z_j^{w,\alpha}$ for all j except of such j that $q_{j-1}^{\mathbf{w}} < \alpha \le q_j^{\mathbf{w}}$. Moreover, having such j, $y_i^{w,\alpha} = 1$ for all $i \le j$ and $y_i^{w,\alpha} = 0$ else. Similarly, $z_i^{w,\alpha} = 1$ for all $i \le j-1$ and $z_i^{w,\alpha} = 0$ else. Therefore $-\mathbf{x}^1\mathbf{w}y_1^{w,\alpha} + \sum_{j=2}^{T}(-\mathbf{x}^j\mathbf{w} + \mathbf{x}^{j-1}\mathbf{w})y_j^{w,\alpha} = -\mathbf{x}^i\mathbf{w}$ for i such that $q_{i-1}^{\mathbf{w}} < \alpha \le q_i^{\mathbf{w}}$.

Since the returns are ordered, $-\mathbf{x}^i\mathbf{w} = (-X\mathbf{w})^{[i]}$, and the rest of the proof directly follows from Definition 1.

Applying Theorem 3 to $VaR_{q_s^{\mathbf{v}}}(-rw)$ is straightforward. However, before applying it to calculation of $\mathrm{VaR}_{q_s^{\mathbf{v}}}(-\mathbf{rv})$ and $\mathrm{VaR}_{q_s^{\mathbf{w}}}(-\mathbf{rv})$ one must reorder the losses of portfolio $\mathbf{v}$ again in ascending order. It can be done by a permutation matrix $M = \{m_{t,j}\}_{t,j=1}^{T}$ and auxiliary vector $\mathbf{k} = (k_1, \dots, k_T)$' such that $-MX\mathbf{v} = \mathbf{k}$ and $k_1 \le k_2 \le \cdots \le k_T$. Moreover the corresponding probabilities, can be obtained as $M\mathbf{p}$. Having this we can again apply Theorem 3 where k_j plays the role of $-\mathbf{x}^j\mathbf{w}$. Summarizing (1) can be rewritten as the following mixed-integer nonlinear problem:

$$\varphi(\mathbf{w}) = \min_{a_s, b_s, \mathbf{v}} \sum_{s=1}^{T} a_s + b_s$$

s.t.

$$\begin{gathered}
\mathbf{k} + \mathbf{x}^1\mathbf{w}y_1^{w,s} + \textstyle\sum_{j=2}^{T}(\mathbf{x}^j\mathbf{w} - \mathbf{x}^{j-1}\mathbf{w})y_j^{w,s} \le a_s,\ s=1,2,\dots,T \\
k_1 y_1^{\mathbf{v},s} + \textstyle\sum_{j=2}^{T}(k_j - k_{j-1})y_j^{\mathbf{v},s} + \mathbf{x}^i\mathbf{w} \le b_s,\ s=1,2,\dots,T \\
-MX\mathbf{v} = \mathbf{k} \\
k_j \le k_{j+1},\ j = 1,2,\dots,T-1 \\
\textstyle\sum_{t=1}^{T} m_{t,j} = 1,\ j = 1,2,\dots,T \\
\textstyle\sum_{j=1}^{T} m_{t,j} = 1,\ t = 1,2,\dots,T \\
\textstyle\sum_{j=1}^{T} y_j^{w,s} p_j \ge q_s^{\mathbf{v}},\ \sum_{j=1}^{T} z_j^{w,s} p_j \le q_s^{\mathbf{v}},\ \sum_{j=1}^{T} z_j^{w,s} + 1 = \sum_{j=1}^{T} y_j^{w,s},\ s=1,2,\dots,T \\
y_j^{w,s} \ge y_{j-1}^{w,s},\ z_j^{w,s} \ge z_{j-1}^{w,s},\ j = 2,\dots,T,\ s = 1,2,\dots,T \\
\textstyle\sum_{j=1}^{T} y_j^{\mathbf{v},s} p_j^{\mathbf{v}} \ge q_s^{\mathbf{w}},\ \sum_{j=1}^{T} z_j^{w,s} p_j^{\mathbf{v}} \le q_s^{\mathbf{w}},\ \sum_{j=1}^{T} z_j^{w,s} + 1 = \sum_{j=1}^{T} y_j^{w,s},\ s=1,2,\dots,T \\
y_j^{\mathbf{v},s} \ge y_{j-1}^{\mathbf{v},s},\ z_j^{\mathbf{v},s} \ge z_{j-1}^{\mathbf{v},s},\ j = 2,\dots,T,\ s = 1,2,\dots,T \\
p_t^{\mathbf{v}} = \textstyle\sum_{j=1}^{T} m_{t,j} p_j,\ t = 1,2,\dots,T \\
q_s^{\mathbf{v}} = \textstyle\sum_{t=1}^{s} p_t^{\mathbf{v}} \\
q_s^{\mathbf{w}} = \textstyle\sum_{t=1}^{s} p_t \\
a_s, b_s \le 0,\ s=1,2,\dots,T \\
y_j^{w,s}, z_j^{w,s}, y_j^{\mathbf{v},s}, z_j^{\mathbf{v},s}, m_{t,j} \in \{0,1\},\ j,s,t = 1,2,\dots,T \\
\mathbf{v} \in W
\end{gathered}$$

Acknowledgement The research was supported by the Czech Science Foundation (GAČR) under the project No. 402/12/G097.

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Summary

Tento článek se zabývá testováním eficience portfolia vzhledem ke stochastické dominanci prvního řádu. Využitím reformulace relace stochastické dominance ve tvaru porovnání dvou VaR na různých hladinách, práce připomíná obecnou nutnou a postačující podmínku eficience portfolia vzhledem ke stochastické dominanci prvního řádu, která platí za předpokladu obecného diskrétního rozdělení (Dupačová a Kopa 2012). V případě, že scénáře tohoto rozdělení jsou stejně pravděpodobné, tento obecný test se zjednoduší a je ekvivalentní Kuosmanen (2004) testu. Pokud scénáře nejsou stejně pravděpodobné, situace se komplikuje, a proto tato práce odvozuje novou, výpočetně dosažitelnou formulaci tohoto obecného testu, která se opírá o úlohu celočíselného nelineárního programování.